

# On regular configurations and disjoint cycles in shift graphs

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## Abstract

Configurations are necklaces with prescribed numbers of red and black beads. Among all possible configurations, the regular one plays an important role in many applications. In this paper, several aspects of regular configurations are discussed, including construction, uniqueness, symmetry group and the link with balanced words.

Another model of configurations is the polygons formed by a given number of sides of two different lengths. In this context, regular configurations are used to obtain a lower bound for the cycles packing number of shift graphs, a subclass of the directed circulant graphs.

## 1 Introduction

Configurations are necklaces with prescribed numbers of red beads and black beads. More precisely, they are circular arrangements of a fixed number of red and black beads. Besides this model, configurations can also be interpreted in many other ways, such as finite words in symbolic dynamics [17], line drawing in computer graphics [11] and the Kawasaki-Ising model in statistical mechanics.

The main interest of this paper is a class of extremal configurations which we call regular configurations. Intuitively, in these configurations the colors are evenly distributed. Alternatively, they are the closest configurations to the “random” ones. Regular configurations are closely linked with balanced words, a well studied object in symbolic dynamics.

In this paper we will show that there is a unique regular configuration (up to cyclic shifts) for a given number of red beads and black beads. Furthermore, an algorithm for computing this configuration is presented. Some other properties of regular configurations are also discussed. To this end we introduce the concept of dual configurations and prove that a configuration is regular if and only if its dual configuration is regular.

Another topic in this paper is shift graphs. A shift graph is a directed Cayley graph of  $\mathbb{Z}_n$  with two generators. It is also called double loop, cyclic graph or chordal ring in the literature [10, 15, 2, 18]. Shift graphs form a subclass of circulant graphs, a type of graph that has been intensively studied [3, 8, 9, 16, 21] and has a vast number of applications to telecommunication network, VLSI design and distributed computations [2, 5, 14, 18].

For shift graphs, we use regular configurations to obtain a lower bound for the cycles packing number, the maximum number of pairwise vertex disjoint cycles. This result also gives an extremal result for sequences and is used to study the guessing number of shift graphs [7].

The remainder of this paper is organized as follows. In Section 2 we present a few models to represent configurations. The definition of regular configuration is given in Section 3, where we also briefly discuss some of its properties. The notion of dual configurations is introduced in Section 4, with which we study several aspects of regular configurations in Section 5, including construction, uniqueness and the symmetry group. The link between regular configurations and balanced words is discussed in Section 6. The application of regular configurations in shift graphs is investigated in Section 7.

## 2 Configurations and models

Given  $a, b \in \mathbb{N}$ , let  $\text{CONF}(a, b)$  denote the set formed by all configurations with a pair of parameters  $(a, b)$ . Intuitively, in *necklace model*,  $a$  and  $b$  are respectively denoting the number of red beads and black beads. And we will use  $n = a + b$  to denote the length of necklaces. For instance, Fig 2 is a necklace representing a configuration in  $\text{CONF}(6, 4)$ . In this section, we are going to introduce several other models that can represent the configurations in  $\text{CONF}(a, b)$ .

The second model is *the Kawasaki-Ising model*. In this model, a configuration in  $\text{CONF}(a, b)$  is represented by a map  $\phi$  from  $V(C_n)$  to  $\{+, -\}$  such that  $|\phi^{-1}(+)| = a$ . Here  $C_n$  is the cycle graph on  $n$  vertices. Since a necklace in  $\text{CONF}(a, b)$  can be regarded as a vertex coloring of  $C_n$  such that  $a$  vertices are colored with red while the others with black, by denoting red color by  $+$

and black color by  $-$  we can convert a necklace into a map in this model. See Fig 4 for such an example, which is obtained from the necklace in Fig 2.

In some context, it is convenient to represent  $+$  by 1 and  $-$  by 0. Then the image of a map  $\phi$  in  $\text{CONF}(a, b)$ , written as a sequence  $\phi(0)\phi(1)\cdots\phi(n-1)$ , is a word over  $\{0, 1\}$ . More precisely, in *word model*, a configuration in  $\text{CONF}(a, b)$  is represented by a word of length  $n$  and weight  $a$ . Here the weight of a word  $w$  is defined to be the number of 1s in it.

Another model is *line model*, arising in compute graphics to answer the following problem: how to draw a zig-zag line from  $(0, 0)$  to  $(a, b)$  on the screen to approximate the “real” line through these two points [11]. We should notice that the scree is represented by the integer lattice  $\mathbb{Z}^2$  and one step from  $(x, y)$  is either  $(x+1, y)$  or  $(x, y+1)$ . As in Fig 3, each configuration can be represented by such a zig-zag line.

The last model we will mention is *polygon model*, which plays an important role in Section 7. In this model, a red bead is represented by a type I side, a side of length  $\alpha$ , while a black bead by a type II side, a side of length  $\beta$ . Here we always assume  $\alpha \neq \beta$ . Then a configuration in  $\text{CONF}(a, b)$  is a polygon formed by  $a$  type I sides and  $b$  type II sides. See Fig 1 for a configuration in  $\text{CONF}(6, 4)$  with  $\alpha = 1$  and  $\beta = 3$ .

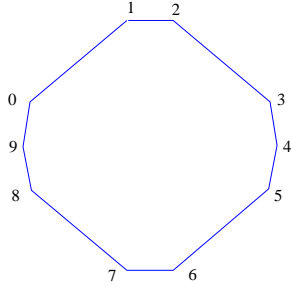


Figure 1: Polygon Model

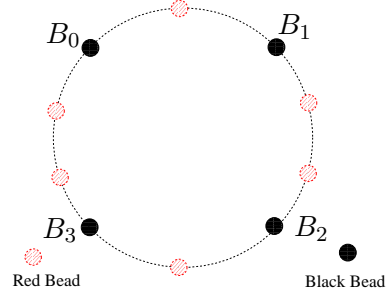


Figure 2: Necklace Model

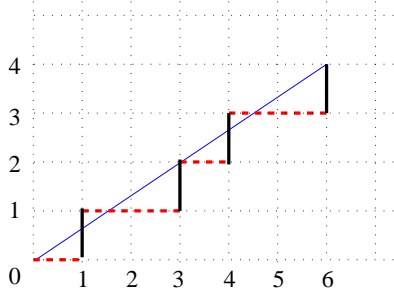


Figure 3: Line Model

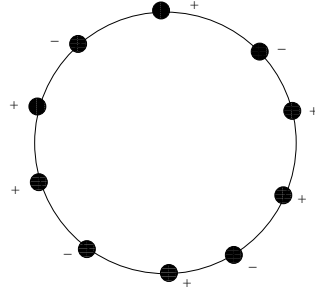


Figure 4: Ising Model

Here we list all these models for  $\text{CONF}(a, b)$  because each model relates configurations to different objects. Thus we can study the structure of  $\text{CONF}(a, b)$  from different views. But in this paper we are mainly focused on necklace model, with some applications related with polygon model and word model. In another paper [6], we study regulations from the view of the Kawasaki-Ising model.

For later use, we will associate two labellings on beads in a necklace. The first one is to label all beads consecutively from 0 to  $n - 1$ . The second one is to consecutively label red(black) beads from 0 to  $a - 1$  ( $b - 1$ ). To avoid potential confusing, in the second labelling the red (black) bead with label  $i$  will be denoted by  $R_i$  ( $B_i$ ). For the first kind labelling on polygons, we assume the vertex is labelled with the number assigned to its right side.

A necklace can be obtained by putting  $b$  black beads in a round and then inserting a certain amount of red beads between each pair of consecutive black beads. More precisely, a configuration  $\Delta \in \text{CONF}(a, b)$  can be represented by a sequence:

$$\Delta = \{B_0, \underbrace{R, \dots, R}_{x_0}, B_1, \underbrace{R, \dots, R}_{x_1}, \dots, B_{b-1}, \underbrace{R, \dots, R}_{x_{b-1}}\}. \quad (1)$$

where  $x_i$  is the number of red beads between black beads  $B_i$  and  $B_{i+1}$ . For brevity, we also say the sequence  $\{x_0, \dots, x_{b-1}\}$  is the *characteristic sequence* of the configuration  $\Delta$ . For instance,  $\{1, 2, 1, 2\}$  is the characteristic sequence of the configuration in Fig 2.

Two necklaces are considered the same if we can cyclicly rotate one to the other. To formulate this in the sequence level, we need the following definitions.

**Definition 2.1.** A shift operator  $\sigma$  on a sequence  $\{x_0, \dots, x_{b-1}\}$  is defined to be:  $\sigma\{x_0, \dots, x_{b-1}\} = \{x_1, \dots, x_{b-1}, x_0\}$ .

**Definition 2.2.**  $\{x_0, \dots, x_{b-1}\} \sim \{x'_0, \dots, x'_{b-1}\}$  if and only if there exists an integer  $t \in [0, b - 1]$  such that  $\{x'_0, \dots, x'_{b-1}\} = \sigma^t\{x_0, \dots, x_{b-1}\}$  where  $\sigma^t$  means applying the operator  $\sigma$  on the sequence for  $t$  times.

In other words,  $\sim$  is an equivalence relation. Furthermore,  $\{x_0, \dots, x_{b-1}\}$  and  $\{x'_0, \dots, x'_{b-1}\}$  are in the same equivalent class if and only if there exists an integer  $t \in [0, b - 1]$  such that  $x_i = x'_{i+t}$  for all  $i$ . Here the index of the elements in the sequence is calculated with modulo the length of the sequence. The same convenience will apply to the other sequences in this paper.

**Proposition 2.1.**  $\{x_0, \dots, x_{b-1}\}$  and  $\{x'_0, \dots, x'_{b-1}\}$  characterize the same configuration in  $\text{CONF}(a, b)$  if and only if  $\{x_0, \dots, x_{b-1}\} \sim \{x'_0, \dots, x'_{b-1}\}$ .

The above proposition can be verified directly from the definitions. Intuitively, it says that a configuration is characterized by a unique equivalence class of sequences. Therefore, in the following sections we also use a sequence to represent a configuration in  $\text{CONF}(a, b)$ .

### 3 Regular configurations

Among all possible configurations in  $\text{CONF}(a, b)$ , the regular one plays an important role in many applications. In this section we will give a precise definition of regularity and study some of its properties. When  $a = 0$  or  $b = 0$ , there is only one configuration in  $\text{CONF}(a, b)$ , the necklace formed by all black beads or by all red beads. To avoid this trivial case, in the remainder of this paper, we will assume  $a > 0$  and  $b > 0$  unless explicitly stated otherwise.

Given a configuration  $\Delta \in \text{CONF}(a, b)$  with its characteristic sequence  $\{x_0, \dots, x_{b-1}\}$ , the following equation holds since each red bead should be inserted between a pair of consecutive black beads.

$$x_0 + x_1 + \dots + x_{b-1} = a \quad (2)$$

Intuitively, when  $a = bt$  for some  $t \in \mathbb{N}$ , the regular configuration is characterized by the sequence  $\{t, t, \dots, t\}$ . In fact, in this case  $t$  is the expected number of red beads between each pair of consecutive black beads in a random configuration. By random configuration we mean to put the beads independently and randomly along the circle. But when  $a/b$  is not an integer, we need the following more subtle condition. The characteristic sequence of a regular configuration optimizes the following problems:

$$\text{Min}(|(x_i + \dots + x_{i+k-1}) - k\frac{a}{b}|), \forall i, k. \quad (3)$$

The above expressions measure the deviation between two quantities: the left one is  $(x_i + \dots + x_{i+k-1})$ , the number of red beads between  $B_i$  and  $B_{i+k}$  in the configuration, and the right one is the expected number of red beads between  $B_i$  and  $B_{i+k}$  in a random configuration. The smaller value of this deviation (or *discrepancy* as it is sometimes called) would imply the configuration is closer to the random one. Here we should notice that the deviation is measured for all possible  $i$  and  $k$ .

As  $(x_i + \dots + x_{i+k-1})$  is always an integer, the expressions in (3) can be simplified as:

$$\frac{a}{b}k - 1 < x_i + x_{i+1} + \dots + x_{i+k-1} < \frac{a}{b}k + 1 \quad (4)$$

for  $0 \leq i \leq b-1, 1 \leq k \leq 1 + \lfloor \frac{b}{2} \rfloor$ . Here we only need to consider the cases for  $1 \leq k \leq 1 + \lfloor \frac{b}{2} \rfloor$  as the other cases can be reduced from them via Equation (2).

Denote the system of inequalities in (4) by  $\text{Reg}(a, b)$ . Then a sequence satisfies  $\text{Reg}(a, b)$  if and only if all its equivalent sequences satisfy it. With these preparations, we are ready to present the formal definition of regularity.

**Definition 3.1.** A configuration  $\Delta$  in  $\text{CONF}(a, b)$  is regular if its characteristic sequences satisfy the inequalities  $\text{Reg}(a, b)$ . In this case, we also say its characteristic sequences are regular.

Let  $\mu_j$  be the minimal number of red beads among  $j+1$  consecutive black beads. More precisely, let  $\mu_{-1} = \mu_0 = 0$  and

$$\mu_j = \min_{0 \leq i \leq b-1} \{x_i + x_{i+1} + \cdots + x_{i+j-1}\}$$

for  $0 < j \leq b$ . Then the regularity can be characterized in the following way:

**Lemma 3.1.**  $\Delta$  is regular if and only if  $1 + \mu_j > \frac{a}{b}j$  for  $-1 \leq j \leq b$ .

*Proof.* “ $\Rightarrow$ ” This direction can be verified directly from the inequalities in (4). “ $\Leftarrow$ ” In this direction, the left inequalities in (4) are easy. From the assumptions and Equation (2), we have:

$$x_i + x_{i+1} + \cdots + x_{i+k-1} \leq a - \mu_{b-k} < a - \left(\frac{a}{b}(b-k) - 1\right) = 1 + \frac{a}{b}k$$

for  $0 \leq i \leq b-1, 1 \leq k \leq 1 + \lfloor \frac{b}{2} \rfloor$ . This completes the right ones.  $\square$

The following two propositions can be verified directly from the above lemma. Herew we use the fact that there exists a unique regular configuration in  $\text{CONF}(a, b)$ , which will be proved in Section 5

**Proposition 3.2.** When  $a = bt$ , the regular configuration in  $\text{CONF}(a, b)$  is characterized by the sequence  $\{t, t, \dots, t\}$ .

**Proposition 3.3.** When  $a = bt + 1$ , the regular configuration in  $\text{CONF}(a, b)$  is characterized by the sequence  $\{t+1, t, \dots, t\}$ .

Intuitively, inserting an equal amount of red beads between each pair of consecutive black beads will not affect the regularity of the original configuration. This can be stated explicitly as the following proposition.

**Proposition 3.4.** Let  $a = tb + r$ . The regular configuration in  $\text{CONF}(a, b)$  is characterized by the sequence  $\{x_0, \dots, x_{b-1}\}$  if and only if the regular configuration in  $\text{CONF}(r, b)$  is characterized by the sequence  $\{x'_0, \dots, x'_{b-1}\}$  with  $x'_j = x_j - t$ .

*Proof.* We will prove one direction and leave the other as an exercise to the readers. From Lemma 3.1, we have  $1 + \mu_j > \frac{a}{b}j$ , which implies

$$1 + \mu'_j = 1 + \mu_j - tj > \frac{a}{b}j - tj = \frac{bt+r}{b}j - tj = \frac{r}{b}j$$

for  $-1 \leq j \leq b$ . By Lemma 3.1, this means that  $\{x'_0, \dots, x'_{b-1}\}$  is regular.  $\square$

## 4 Dual configurations

In this section we will introduce the concept of duality and prove that one configuration is regular if and only if its dual configuration is regular.

Let  $\Delta$  be a configuration in  $\text{CONF}(a, b)$  characterizing by  $\{x_0, \dots, x_{b-1}\}$ . More precisely,  $\Delta$  can be expressed in the following form:

$$\Delta = \{B_0, \underbrace{R, \dots, R}_{x_0}, B_1, \underbrace{R, \dots, R}_{x_1}, \dots, B_{b-1}, \underbrace{R, \dots, R}_{x_{b-1}}\}.$$

From it we can construct a new configuration:

$$\Delta^* = \{R_0^*, \underbrace{B^*, \dots, B^*}_{x_0}, R_1^*, \underbrace{B^*, \dots, B^*}_{x_1}, \dots, R_{b-1}^*, \underbrace{B^*, \dots, B^*}_{x_{b-1}}\}.$$

Intuitively,  $\Delta^*$  is obtained from  $\Delta$  by switching the colors of the beads. More explicitly,  $B_i^*$ , a black bead in  $\Delta^*$ , is obtained from  $R_i$  in  $\Delta$  and  $R_j^*$  is from  $B_j$ . Then  $\Delta^*$  belongs to  $\text{CONF}(b, a)$  and it can be expressed in the following way.

$$\Delta^* = \{B_0^*, \underbrace{R^*, \dots, R^*}_{y_0}, B_1^*, \underbrace{R^*, \dots, R^*}_{y_1}, \dots, B_{a-1}^*, \underbrace{R^*, \dots, R^*}_{y_{a-1}}\}.$$

Here  $y_i$  in the above representation is the number of red beads between  $B_i^*$  and  $B_{i+1}^*$  in  $\Delta^*$ , which is equal to the number of black beads between  $R_i$  and  $R_{i+1}$  in  $\Delta$  from the construction. See Fig 5 for the dual configuration of that in Fig 2.

Since the color on each bead will remain the same after twice switching, we have the following proposition.

**Proposition 4.1.**  $(\Delta^*)^* = \Delta$ .

Recall that the regular condition,  $\text{Reg}(b, a)$ , for the configurations in  $\text{CONF}(b, a)$  is:

$$\frac{b}{a}t - 1 < y_j + y_{j+1} + \dots + y_{j+t-1} < \frac{b}{a}t + 1 \quad (5)$$

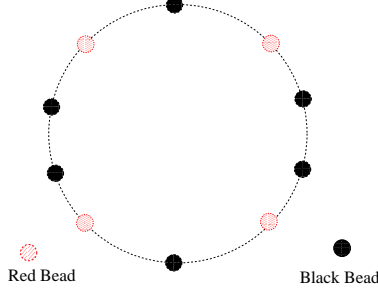


Figure 5: A dual configuration

for all  $0 \leq j \leq a-1$  and  $1 \leq t \leq 1 + \lfloor \frac{a}{2} \rfloor$ .

The following lemma is the main result of this section, which plays an important role in the remainder of this paper.

**Lemma 4.2.** *A configuration in  $\text{CONF}(a, b)$  is regular if and only if its dual configuration, which belongs to  $\text{CONF}(b, a)$ , is regular.*

*Proof.* By Lemma 4.1, it is enough to show one direction.

Given a configuration  $\Delta$  in  $\text{CONF}(a, b)$  characterizing by the sequence  $\{x_0, \dots, x_{b-1}\}$ , we need to show  $\{y_0, \dots, y_{a-1}\}$ , which characterizes the dual configuration  $\Delta^*$ , satisfies  $\text{Reg}(b, a)$ . That means for all  $0 \leq j \leq a-1, 1 \leq t \leq 1 + \lfloor \frac{a}{2} \rfloor$ , the following inequality holds.

$$\frac{b}{a}t - 1 < y_j + y_{j+1} + \dots + y_{j+t-1} < \frac{b}{a}t + 1. \quad (6)$$

Without loss of generality, we can prove the above inequalities for  $j = 1$ . Denote the number of red beads between  $B_1^*$  and  $B_{t+1}^*$  by  $\tau$ . In other words,  $\tau = y_1 + \dots + y_t$ . We will prove the lemma by considering three different cases of  $\tau$ .

**Case 1:**  $\tau = 0$ . In this case there is no red bead between  $B_1^*$  and  $B_{t+1}^*$ . That means there is no black bead between  $R_1$  and  $R_{t+1}$  in  $\Delta$ , which implies the red beads with labelling from  $R_1$  to  $R_{t+1}$  are all falling between  $B_i$  and  $B_{i+1}$  for some  $i$ . Thus  $t+1 < x_i$  for some  $i$ , from which we have

$$\begin{aligned} t+1 < x_i \text{ for some } i &\Rightarrow t+1 < 1 + \frac{a}{b} \\ &\Rightarrow t < \frac{a}{b} \\ &\Rightarrow \frac{b}{a}t - 1 < y_1 + y_2 + \dots + y_t = 0 < \frac{b}{a}t + 1, \end{aligned}$$



where the first line comes from the regularity of  $\{x_0, \dots, x_{b-1}\}$ .

**Case 2:**  $\tau = 1$ . Similar to the proof in case 1, we know there is only one black bead between  $R_1$  and  $R_{t+1}$  in  $\Delta$ , which means  $t+1 < x_i + x_{i+1}$  for some  $i$ . Thus we have:

$$\begin{aligned} t+1 < x_i + x_{i+1} \text{ for some } i &\Rightarrow t+1 < 1 + 2\frac{a}{b} \\ &\Rightarrow \frac{b}{a}t - 1 < 1 \\ &\Rightarrow \frac{b}{a}t - 1 < y_1 + y_2 + \dots + y_t = 1 < \frac{b}{a}t + 1. \end{aligned}$$

**Case 3:**  $\tau \geq 2$ . Now we can assume that there are  $k+1$  red beads between  $B_1^*$  and  $B_{t+1}^*$  for some  $k \geq 1$ . In other words, there are  $k+1$  black beads between  $R_1$  and  $R_{t+1}$  in  $\Delta$ . Assume these black beads are labelled from  $B_i$  to  $B_{i+k}$ . Then we have the following fragment in the sequence representing  $\Delta$ :

$$R_1, \underbrace{R, \dots, R}_{\epsilon_1}, B_i, \dots, B_{i+k}, \underbrace{R, \dots, R}_{\epsilon_2}, R_{t+1}$$

where  $0 \leq \epsilon_1 \leq x_{i-1} - 1$  and  $0 \leq \epsilon_2 \leq x_{i+k} - 1$ .

Since there are  $t-1$  red beads between  $R_1$  and  $R_{t+1}$ , we have:

$$(\epsilon_1 + 1) + (\epsilon_2 + 1) + (x_i + \dots + x_{i+k-1}) = (t-1) + 2 = t+1. \quad (7)$$

From the regularity of  $\Delta$ ,  $x_i + \dots + x_{i+k-1} > k\frac{a}{b} - 1$ . Putting it into Equation (7) and noting that  $\epsilon_1 \geq 0$  and  $\epsilon_2 \geq 0$ , we have:

$$t+1 > k\frac{a}{b} - 1 + 2. \quad (8)$$

On the other hand, from  $\epsilon_1 \leq x_{i-1} - 1$  and  $\epsilon_2 \leq x_{i+k} - 1$ , Equation (7) implies:

$$t+1 \leq x_{i-1} + x_i + \dots + x_{i+k} < \frac{a}{b}(k+2) + 1. \quad (9)$$

Put (8) and (9) together, we have:

$$\begin{aligned} k\frac{a}{b} - 1 + 2 &< t+1 < \frac{a}{b}(k+2) + 1 \\ \Rightarrow k\frac{a}{b} &< t < \frac{a}{b}(k+2) \\ \Rightarrow t\frac{a}{b} - 2 &< k < t\frac{a}{b} \\ \Rightarrow t\frac{a}{b} - 1 &< k+1 < t\frac{a}{b} + 1, \end{aligned}$$

which completes the proof of the last case in the lemma since  $y_1 + y_2 + \dots + y_t = k+1$ .  $\square$

To summarize, in this section we prove  $*$ , the dual operator, maps each configuration  $\Delta$  in  $\text{CONF}(a, b)$  to a configuration  $\Delta^*$  in  $\text{CONF}(b, a)$ . Furthermore, this map is an onto bijection and preserves the regularity.

## 5 Construction and symmetry

In this section, two aspects of regular configuration are discussed. The first one is the existence of a unique regular configuration in  $\text{CONF}(a, b)$ . The other is the symmetry group of regular configurations.

### 5.1 Construction

An algorithm to construct a regular configuration in  $\text{CONF}(a, b)$  is proposed in this subsection. In fact, there exists another well known algorithm in computer graphics for this problem [11]. But the one presented here is more convenient for our propose. Furthermore, we will prove such regular configuration in  $\text{CONF}(a, b)$  is unique.

The input of the algorithm is  $\delta: (a, X; b, Y)$ . Here  $X$  ( $Y$ ) is a fragment of necklaces;  $a$  and  $b$  are respectively the number of  $X$  fragments and  $Y$  fragments. The output is  $\Delta$ , a necklace formed by  $a$   $X$  fragments and  $b$   $Y$  fragments. As in the previous sections, the output configuration  $\Delta$  also will be represented by its characteristic sequence.

FindRegular  $(a, X; b, Y)$

- if  $(a < b)$ , return FindRegular( $b, Y; a, X$ );
- else do:
  - compute  $t, k$  such that  $a = bt + k$  where  $1 \leq t, 0 \leq k < b$ .
  - build a new fragment  $Z = \{Y, X, \dots, X\}$  with  $t$  fragments of  $X$ .
    - \* if  $(k \neq 0)$  return FindRegular( $b, Z; k, X$ );
    - \* else return a necklace formed by  $b$  fragments of  $Z$ .

**Algorithm I: Find a regular configuration**

When the fragment  $X$  is only a red bead and  $Y$  is a black bead, the input parameters  $(a, X; b, Y)$  can be simplified as  $(a, b)$ . In this case, we will prove

that the output configuration, which belongs to  $\text{CONF}(a, b)$ , is regular. To this end, we consider some special cases.

**Proposition 5.1.** *For the input  $\delta_1 = (bt, b)$ , the output configuration  $\Delta_1$  is given by the characteristic sequences  $\{t, \dots, t\}$ . Similarly, for  $\delta_2 = (bt+1, b)$ ,  $\Delta_2$  is characterized by  $\{t+1, t, \dots, t\}$ . In both cases, the output configurations are regular.*

*Proof.* For  $i = 1, 2$ , it can be verified directly that  $\Delta_i$  is the output configuration for  $\delta_i$ . Furthermore  $\Delta_i$  is regular from Proposition 3.2 and 3.3.  $\square$

**Proposition 5.2.** *Let  $a = tb + r$  for nonnegative integers  $t$  and  $r$ . If  $\{x_0, \dots, x_{b-1}\}$  characterizes the output configuration of  $\text{FindRegular}(a, b)$ , then  $\{x'_0, \dots, x'_{b-1}\}$ , where  $x'_j = x_j - t$ , characterizes the output configuration of  $\text{FindRegular}(r, b)$ .*

*Proof.* The proposition can be verified by comparing the outputs of the algorithm for inputs  $(bt + r, b)$  and  $(r, b)$ .  $\square$

**Proposition 5.3.** *If  $\Delta$  is the output of  $\text{FindRegular}(a, b)$ , then  $\Delta^*$ , the dual configuration of  $\Delta$ , is the output of  $\text{FindRegular}(b, a)$ .*

*Proof.*  $\Delta^*$  can be obtained from  $\Delta$  by switching the color of all beads. This process can be done either before running the algorithm to get  $\Delta$  or after running it. In the first case, it is the same to say the input is  $(b, a)$ .  $\square$

With these preparations we can prove the following theorem, which is the main result of this subsection.

**Theorem 5.4.** *Given any integer pair  $(a, b)$  as the input, the output of the algorithm  $\text{FindRegular}$  is always a regular configuration.*

*Proof.* We will prove this theorem by induction on  $b$ .

**Step 1:** The base case is  $a = bt$ , which also contains  $b = 1$ . In this case the theorem holds from Proposition 5.1.

**Step 2:** Now let  $(a, b)$  be an instance of the input such that  $b$  is smallest over all instances that the output is a irregular configuration. From step 1, we can assume  $a = tb + r$  for some integer  $t$  and  $r$  where  $0 < r < b$ . From Proposition 5.2 and 3.4 the output for  $(r, b)$  is also irregular. Furthermore, from Proposition 5.3 and Lemma 4.2, the output for  $(b, r)$  is also irregular, which contradicts the minimality of  $b$ .  $\square$

The above theorem implies the existence of a regular configuration in  $\text{CONF}(a, b)$ . Now we are going to show that such regular configuration is in fact unique in  $\text{CONF}(a, b)$ .

**Theorem 5.5.** *There exists at most one regular configuration in  $\text{CONF}(a, b)$ .*

*Proof.* The theorem holds for  $b = 0$  and  $b = 1$  since in both cases there is essentially one configuration and it is regular by the definition.

Now assume the theorem fails for some  $b$  and let  $b$  be the smallest one such that  $\text{CONF}(a, b)$  contains two different regular configurations  $\Delta, \Delta'$  for some  $a$ , where  $\Delta$  and  $\Delta'$  are characterized respectively by two non-equivalent sequences  $x_0, \dots, x_{d-1}$  and  $z_0, \dots, z_{d-1}$ . Now we have  $a = bt + r$  for some  $t \in \mathbb{N}$  and integer  $0 \leq r < b$ . From proposition 3.4 we know  $x'_0, \dots, x'_{d-1}$  and  $z'_0, \dots, z'_{d-1}$  represent two different regular configurations in  $\text{CONF}(r, b)$ . That implies  $\text{CONF}(b, r)$  contains two different regular configurations from Lemma 4.2, a contradiction to the minimality of  $b$ .  $\square$

We summarize Theorem 5.4 and 5.5 as the following one.

**Theorem 5.6.** *There exists a unique regular configuration in  $\text{CONF}(a, b)$ , which can be constructed by Algorithm I.*

## 5.2 The symmetry group

In this subsection we assume the necklaces in  $\text{CONF}(a, b)$  is given with the labelling of the first type. That means the beads are consecutively labelled from 0 to  $n - 1$  with  $n = a + b$ . In this case, we also call such a necklace, together with its labelling, as a *labelled necklace*. Furthermore, the bead in a labelled necklace  $\Delta$  will be denoted by  $t_i$ .

Now two labelled necklaces are essentially same if we can cyclically permute one to the other. In other words, they correspond to the same (unlabelled) necklace. More precisely, we have the following definition.

**Definition 5.1.** Given an integer  $k \in [0, n - 1]$ , the rotation  $\phi_k$ , which is defined as  $\phi_k(i) = i + k \pmod{n}$  for  $i \in [0, n - 1]$ , is called a cyclic permutation of the labelled necklace  $\Delta$  in  $\text{CONF}(a, b)$  if  $t_i$  and  $t_{\phi_k(i)}$  have the same color for each  $i$ .

All cyclically permutations of  $\Delta$  form a group, called the *symmetry group* of  $\Delta$  and denoted by  $\text{Rot}(\Delta)$ . Notice that two labelled necklaces  $\Delta$  and  $\Delta'$  are the same if  $t_i = t'_i$  for each  $i$ .

**Proposition 5.7.** *There are exactly  $(a + b)/|\text{Rot}(\Delta)|$  different labelled configurations associated with the same unlabelled configuration  $\Delta$ .*

*Proof.* Given an unlabelled necklace  $\Delta$ , we can assign it with a first type labelling and denote such a labelled necklace as  $\Delta_0$ . From this we can obtain

a set of labelled necklaces  $\{\Delta_0, \dots, \Delta_{n-1}\}$ , via  $t_0^i = t_i^0$ . That is, we build  $\Delta_i$  by assigning 0 to  $t_i$  in  $\Delta_0$ . Now  $\Delta_i$  and  $\Delta_j$  are the same if and only if there exists an element  $\phi \in \text{Rot}(\Delta)$  such that  $\Delta_i = \phi(\Delta_j)$ . Since any labelled necklace obtained from  $\Delta$  must equal to some  $\phi_i$ , we know there are exactly  $(a+b)/|\text{Rot}(\Delta)|$  different labelled configurations corresponding to  $\Delta$ .  $\square$

Intuitively, a configuration  $\Delta \in \text{CONF}(a, b)$  is symmetric if its symmetry group  $\text{Rot}(\Delta)$  has the maximal size over all possible labelled configurations. Note that this maximal number is bounded above by  $\gcd(a, b)$ , the greatest common divisor of  $a$  and  $b$ . This is because each element in  $\text{Rot}(\Delta)$  induces two cyclic permutations, one for a labelled necklace in  $\text{CONF}(a, 0)$  and the other for that in  $\text{CONF}(0, b)$ .

**Definition 5.2.** A configuration  $\Delta$  is symmetric if its symmetry group  $\text{Rot}(\Delta)$  has size  $\gcd(a, b)$ .

**Proposition 5.8.** Let  $a = tb + r$  for nonnegative integers  $t$  and  $r < b$ . Given a configuration  $\Delta$  in  $\text{CONF}(a, b)$  characterized by  $\{x_0, \dots, x_{b-1}\}$ , let  $s$  be the smallest number of  $x_i$ . Then  $\{x'_0, \dots, x'_{b-1}\}$ , where  $x'_j = x_j - s$ , characterizes a configuration  $\Delta'$  in  $\text{CONF}(a - sb, b)$ . Furthermore, their symmetry groups have the same size. That is,  $|\text{Rot}(\Delta)| = |\text{Rot}(\Delta')|$ .

*Proof.* The proposition holds because inserting  $s$  red beads for each consecutive pairs of black beads will extend a cyclic permutation of  $\Delta'$  to that of  $\Delta$ . And all cyclic permutation in  $\text{Rot}(\Delta)$  can be obtained by this way.  $\square$

When  $\Delta$  is the regular configuration in  $\text{CONF}(a, b)$ , from the definition of regularity we know that the  $s$ , which is defined in the above proposition, is equal to  $t$ . In this case, the following corollary holds.

**Corollary 5.9.** Let  $a = tb + r$  for nonnegative integers  $t$  and  $r < b$ . Given the regular configuration  $\Delta = \{x_0, \dots, x_{b-1}\}$  in  $\text{CONF}(a, b)$ , we can construct a configuration  $\Delta'$  in  $\text{CONF}(r, b)$  characterized by  $\{x'_0, \dots, x'_{b-1}\}$ , where  $x'_j = x_j - t$ . Then  $\Delta'$  is regular and  $|\text{Rot}(\Delta)| = |\text{Rot}(\Delta')|$ .

Unlike regular configurations, symmetric configurations in  $\text{CONF}(a, b)$  are not unique. For instance, Fig 1 and Fig 7 show two symmetric configurations in  $\text{CONF}(6, 4)$ . See Fig 6 for an example of nonsymmetric configuration. Here we represent the configurations in the polygon model for better visualization. Therefore, symmetry generally does not imply regularity. But the converse is true, as the following theorem implies.

**Theorem 5.10.** Regular configurations are symmetric.

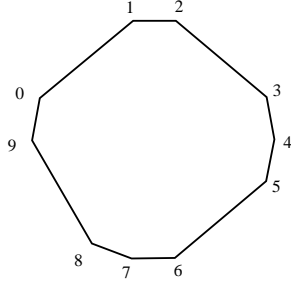


Figure 6: Not symmetric



Figure 7: Symmetric but not regular

*Proof.* Given the regular configuration  $\Delta \in \text{CONF}(a, b)$ , we prove  $|\text{Rot}(\Delta)| = \gcd(a, b)$  by an induction on  $b$ .

**Step 1:** The base case is  $a = bt$ , which includes  $b = 1$ . In this case the regular configuration  $\Delta$  in  $\text{CONF}(a, b)$  is symmetric as  $|\text{Rot}(\Delta)| = t$  from Proposition 5.1 and  $\gcd(a, b) = t$ .

**Step 2:** Now assume the theorem fails for some  $\text{CONF}(a, b)$  and let  $b$  be the smallest one such that the regular configuration  $\Delta$  in  $\text{CONF}(a, b)$  satisfies  $|\text{Rot}(\Delta)| < \gcd(a, b)$ . From step 1, we can assume  $a = bt + r$  for some  $t \in \mathbb{N}$  and  $r \in (0, b)$ . Now consider  $\Delta'$ , the regular configuration in  $\text{CONF}(r, b)$ , which is characterized by  $(x'_0, \dots, x'_{a-1})$  by Proposition 3.4. This implies  $|\text{Rot}(\Delta)| = |\text{Rot}(\Delta')|$  from Corollary 5.9. Since  $\gcd(a, b) = \gcd(r, b)$ ,  $\Delta'$  is not symmetric, a contradiction to the minimality of  $b$ .  $\square$

## 6 Balanced words

In this section we will discuss the relations between balanced words and regular configurations. Let  $\omega = \omega_0\omega_1 \dots \omega_{n-1} \in \{0, 1\}^n$  be a word of length  $n$  over alphabet  $\{0, 1\}$ . The weight of  $\omega$ , denoted by  $|\omega|_1$ , is the number of 1s appeared in  $\omega$ . All words of length  $n$  and weight  $k$ , where  $0 \leq k \leq n$ , form a set  $\mathbb{W}_{k,n}$ .

An operator  $\sigma$ , called *cyclic shift*, is defined on  $\mathbb{W}_{k,n}$  as follows:  $\sigma(w) = w_1 \dots w_{n-1}w_0$ . This gives an equivalence relation on  $\mathbb{W}_{k,n}$ :  $w \sim w'$  if and only if they belong to the same cyclic shifting orbit. Here the *cyclic shifting orbit* of a word  $w$  is defined to be  $\{\sigma^i(w) \mid 0 \leq i < a + b\}$ .

From Section 2, a word  $w \in \mathbb{W}_{a,a+b}$  can be regarded as a labelled configuration  $\Delta \in \text{CONF}(a, b)$ . From Proposition 5.7, each (unlabelled) necklace  $\Delta$  can be associated with  $(a + b)/|\text{Rot}(\Delta)|$  different words, which form an orbit of the cyclic shifting. More precisely, we have the following relation between

words and configurations:

$$\frac{\mathbb{W}_{a,a+b}}{\sim} \cong \text{CONF}(a, b).$$

In the remainder of this section, we are going to show that regular configurations are related to balanced words, an important class of words.

Let  $|w|$  denote the length of  $w$  and define  $|w|_0$  as  $|w| - |w|_1$ . A *cyclic subword* of  $w$  is any length- $q$  prefix of some  $\sigma^{i-1}(w)$  for  $1 \leq i, q \leq m$ . Then we have the following definition.

**Definition 6.1.** A word  $w$  is called *balanced* if for any two of its cyclic subwords  $z$  and  $z'$ ,  $|z| = |z'|$  implies  $||z|_i - |z'|_i| \leq 1$  for  $i = 0, 1$ .

Balanced words, the finite version of sturmian words, form an important class of words. We recommend [17] for more backgrounds and [12, 13] for some recent developments.

**Theorem 6.1.** A configuration  $\Delta \in \text{CONF}(a, b)$  is regular if and only if any of its associated word  $w$  is balanced.

*Proof.* By definition, a word  $w$  is balanced if and only if any word in its cyclic shifting orbit is balanced. Thus we can always choose a convenient one in the orbit corresponding to  $\Delta$  to simplify the following proof. Without loss of generality, we can also assume  $a \geq b$ .

“ $\Rightarrow$ ”: If  $w$  is not balanced, then  $||u|_i - |v|_i| \geq 2$  for a pair of cyclic subwords  $u$  and  $v$  with the same length, say  $t$ . Without loss of generality, we can assume  $|u|_0 - |v|_0 \geq 2$ . Furthermore, we associate a second kind labelling on black beads, i.e., the 0s in the word. Assume the first 0 appeared in  $u$  is labelled with 1. Then the structure of  $u$  can be schematically represented in the following way.

$$u = \underbrace{1, \dots, 1}_{\epsilon_1}, 0_1, \dots, 0_k, \underbrace{1, \dots, 1}_{\epsilon_2}$$

where  $k$  is the number of 0s appeared in  $u$ ,  $0 \leq \epsilon_1 \leq x_0$  and  $0 \leq \epsilon_2 \leq x_k$ . Here  $x_i$  is the number of 1s (red beads) appeared between  $0_i$  ( $B_i$ ) and  $0_{i+1}$   $B_{i+1}$ . Since  $|u| = t$ , these parameters satisfy the following equation.

$$\epsilon_1 + x_1 + \dots + x_{k-1} + \epsilon_2 + k = t. \quad (10)$$

On the other hand, we have the following representation of  $v$ .

$$v = \underbrace{1, \dots, 1}_{\epsilon'_1}, 0_i, \dots, 0_{s+i-1}, \underbrace{1, \dots, 1}_{\epsilon'_2}.$$

where  $s$  is the number of 0s appeared in  $v$ ,  $0 \leq \epsilon'_1 \leq x_{i-1}$  and  $0 \leq \epsilon'_2 \leq x_{i+s-1}$ . Similar to Equation (10), they satisfy:

$$\epsilon'_1 + x_i + \cdots + x_{i+s-2} + \epsilon'_2 + s = t. \quad (11)$$

Since we can solve  $k$  and  $s$  from Equation (10) and (11) respectively, the condition  $k - s \geq 2$ , which comes from  $|u|_0 - |v|_0 \geq 2$ , becomes:

$$(\epsilon'_1 + x_i + \cdots + x_{i+s-2} + \epsilon'_2) - (\epsilon_1 + x_1 + \cdots + x_{k-1} + \epsilon_2) \geq 2. \quad (12)$$

By the constraints of  $\epsilon$  and  $\epsilon'$ , the above equation can be further simplified as:

$$(x_{i-1} + x_i + \cdots + x_{i+s-1}) - (x_1 + \cdots + x_{k-1}) \geq 2. \quad (13)$$

But from the condition that  $\Delta$  is regular, we can obtain an upper bound of the sum in the first parenthesis and a lower bound for that in the second one.

$$(x_{i-1} + x_i + \cdots + x_{i+s-1}) < (s+1)\frac{a}{b} + 1. \quad (14)$$

$$(x_1 + \cdots + x_{k-1}) > (k-1)\frac{a}{b} - 1. \quad (15)$$

The above two bounds give us the following inequality:

$$\begin{aligned} (x_{i-1} + \cdots + x_{i+s-1}) - (x_1 + \cdots + x_{k-1}) &< (s+1)\frac{a}{b} + 1 - [(k-1)\frac{a}{b} - 1] \\ &= (s+2-k)\frac{a}{b} + 2 \\ &\leq 2. \end{aligned}$$

That is a contradiction to Equation (13). In the last step of above inequalities we use the fact that  $s+2 \leq k$ .

“ $\Leftarrow$ ”: In this direction, we need to prove that  $\Delta$  is regular with the assumption that  $w$ , one of its associated words, is balanced. If this fails, then we have:

$$|x_i + \cdots + x_{i+k-1} - k\frac{a}{b}| \geq 1 \quad (16)$$

for some  $i$  and  $k$ . Among all such pairs  $(i, k)$  satisfied the above inequality, we fix one pair  $(i, k)$  such that  $k$  is the minimal. That means either  $x_i + \cdots + x_{i+k-1} \geq 1 + (ka/b)$  or  $x_i + \cdots + x_{i+k-1} \leq -1 + (ka/b)$ . Here we only prove this direction for the first case as the following arguments can be easily modified for the second one.



Firstly we claim there exists one  $j$  such that  $x_j + \cdots + x_{j+k-1} < ka/b$ . If not, then

$$k(x_0 + \cdots + x_{b-1}) = \sum_{s=0}^{b-1} (x_s + \cdots + x_{s+k-1}) \geq ka + 1,$$

a contradiction to the fact  $(x_0 + \cdots + x_{b-1}) = a$ . Thus we have

$$(x_j + \cdots + x_{j+k-1}) - (x_i + \cdots + x_{i+k-1}) \geq 2 \quad (17)$$

because both sums in the parentheses are integer.

Now there exist the following two fragments in the configuration (Recall that 1 stands for read bead  $R$  and 0 stands for black bead  $B$ ):

$$u = 0_i, \underbrace{1, \dots, 1}_{x_i}, 0_{i+1}, \dots, 0_{i+k-1}, \underbrace{1, \dots, 1}_{x_{i+k-1}}, 0_{i+k}$$

and

$$v = 0_j, \underbrace{1, \dots, 1}_{x_j}, 0_{j+1}, \dots, 0_{j+k-1}, \underbrace{1, \dots, 1}_{x_{j+k-1}}, 0_{j+k}.$$

Furthermore, construct a new fragment  $v'$  by choosing the first  $|u| + 1$  bits from  $v$  and deleting  $0_j$ . Then  $|u| = |v'|$  and there exists at least two more 0s in  $u$  than that in  $v'$  since the 0s in  $u$  are labelled from  $i$  to  $i + k$  while the labels of 0s in  $v'$  are falling into the interval  $[j + 1, j + k - 1]$ . In other words,  $||u|_1 - |v'|_1| \geq 2$ . As each fragment can be realized as a cyclic subword of  $w$ ,  $u$  and  $v'$  are two cyclic subwords with the same length but their weight are different greater than 2, a contradiction to the fact that  $w$  is balanced.  $\square$

From the relation between words and configurations, the above theorem implies the following corollary.

**Corollary 6.2.**  $\mathbb{W}_{a,a+b}$  has exactly  $(a + b)/\gcd(a, b)$  balanced words, which form a cyclic shifting orbit that corresponds to the regular configuration in  $\text{CONF}(a, b)$ .

It is already know in [4, 13] that there are precisely  $a + b$  balanced words in  $\mathbb{W}_{a+b}$  if  $a$  and  $b$  are coprime. The above corollary slightly generalizes that result. From Section 5.2 we also know that the orbit corresponding to the regular configuration should has the smallest size.

## 7 Disjoint cycles in shift graphs

In this section we will study the cycles packing number of  $\text{Shift}(n, m)$ , the directed Cayley graph of  $\mathbb{Z}_n$  with two generators  $\{1, m\}$ .

Recall that the vertex set of  $\text{Shift}(n, m)$  is  $\{0, 1, 2, \dots, n-1\}$  and there are two types of edge sets: type I consists of the edge generating by  $\{1\}$ , i.e., the edge has the form  $(i, i+1) \pmod{n}$ ; type II consists of the edge generating by  $\{m\}$ , i.e., the edge has the form  $(i, i+m) \pmod{n}$ . Here  $i$  runs through all vertices. See Fig 8 for  $\text{Shift}(9, 3)$ .

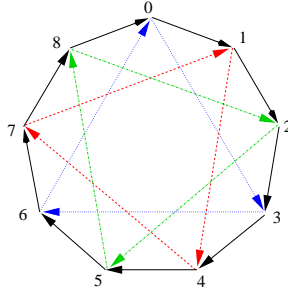


Figure 8: Shift (9,3)

The cycles discussed in this paper are directed. A cycle  $C = v_0v_1 \dots v_{d-1}v_0$  can be represented by its vertex sequence:  $A_C = \langle v_0, v_1, \dots, v_{d-1} \rangle$  where  $d$  is the size of  $C$ . On the other hand, the vertices of a cycle  $C$  is the unordered  $d$ -set:  $V(C) = \{v_0, v_1, \dots, v_{d-1}\}$ .

Two cycles  $C$  and  $C'$  are called (vertex) disjoint if  $V(C) \cap V(C') = \emptyset$ . A collection of disjoint cycles is a set of disjoint cycles  $\mathcal{C} = \{C_1, \dots, C_k\}$  such that they are pairwise disjoint. The size of a collection  $\mathcal{C}$ , is the number of cycles it contains and will be denoted by  $|\mathcal{C}|$ .

**Definition 7.1.** The cycles packing number for a graph  $G$ , denoted by  $\nu_0(G)$ , is defined as:

$$\nu_0(G) = \max\{|\mathcal{C}| \mid \mathcal{C} \text{ is a collection of vertex disjoint cycles in } G\}.$$

Another version of cycles packing number is studying edge disjoint cycles but in this paper we are only considering the vertex disjoint version. We remark that  $\nu_0(G)$  plays an important role in many fields [1]. One recent example is that  $\nu_0(G)$  gives the lower bound of the guessing number of  $G$ , a parameter of graph defined by Riis [7, 19].

Let  $n = am+b$  for  $0 \leq a$  and  $0 \leq b < m$ . Let  $d = a+b$  and  $k = \lfloor n/(a+b) \rfloor$  in the remainder of this section. Then we have the following theorem, which is the main result of this section.

**Theorem 7.1.**  $\nu_0(\text{Shift}(n, m)) \geq k$ . Furthermore, there is an algorithm to produce a collection of disjoint cycles  $\mathcal{C}$  with  $|\mathcal{C}| = k$ .

Before proving the theorem, we use  $\text{Shift}(9, 3)$  (see Fig 8) to illustrate the intuition behind it. Let  $C_1 = \{0, 3, 6\}$ ,  $C_2 = \{1, 4, 7\}$  and  $C_3 = \{2, 5, 8\}$ . Then  $\nu_0(\text{Shift}(9, 3)) = 3$  and  $\mathcal{C} = \{C_1, C_2, C_3\}$  is such a collection. Furthermore,  $C_2$  and  $C_3$  can be regarded as being obtained from  $C_1$  via a “rotation”. Here one crucial observation is that the “polygon” under  $C_1$  is regular. But we need to consider regular configurations for general cases.

To this end, associate a necklace to each cycle in  $\text{Shift}(n, m)$ . Given a cycle  $C$  by its vertex sequence  $A = \langle v_0, v_1, \dots, v_{d-1} \rangle$ . Then its *differential sequence* is defined as  $\langle v_1 - v_0, \dots, v_{d-1} - v_{d-2}, v_0 - v_{d-1} \rangle$  and denoted by  $\nabla(A)$ . Here the subtraction is calculated with modulo  $n$ .

For a cycle  $C$  in  $\text{Shift}(n, m)$ , the sequence  $\nabla(A_C)$  consists of two numbers, 1 and  $m$ . Furthermore, they are respectively corresponding to two types of edges in  $\text{Shift}(n, m)$ . If we represent 1 in  $\nabla(A_C)$  by black bead  $B$  and  $m$  by red bead  $B$ , then  $\nabla(A_C)$  gives us a labelled necklace in a natural way. By forgetting its labelling, we obtain a configuration from  $C$ , which will be denoted by  $\Delta_C$ .

On the other hand, given a configuration  $\Delta$  and a vertex  $v$ , we can construct a path  $C_{v, \Delta}$  in  $\text{Shift}(n, m)$  such that  $C_{v, \Delta}$  contains  $v$  and its associated configuration is  $\Delta$ . Firstly, obtain the differential sequence  $\nabla = \langle l_0, \dots, l_{d-1} \rangle$  from the necklace  $\Delta$ . Then the cycle  $C_{v, \Delta}$ , which will be simplified as  $\underline{v}$  when  $\Delta$  is clear, is given by  $\langle v, v + k_0, v + k_1, \dots, v + k_{d-2} \rangle$  where  $k_p = l_0 + \dots + l_p$ . The cycle  $\underline{0}$ , which will play an important role in the following analysis, is called the *generic cycle* of  $\Delta$  and its vertex set is denoted by  $V_\Delta$ . Now Theorem 7.1 can be restated as the following one.

**Theorem 7.2.** For the regular configuration  $\Delta$  in  $\text{CONF}(a, b)$ , the set  $\{\underline{0}, \underline{(m-1)}, \dots, \underline{(k-1)(m-1)}\}$ , where  $\underline{i} = C_{i, \Delta}$ , is a collection of pairwise disjoint cycles in  $\text{Shift}(n, m)$ .

Notice that we can easily verify the above theorem for  $a = 0$  or  $b = 0$ . In fact, in this case the regular configuration in  $\text{CONF}(n, m)$  is exactly the regular polygon. Therefore, in the following proof, we will assume  $a > 0$  and  $b > 0$  for simplicity. On the other hand, we do not give explicitly the algorithm stated in Theorem 7.1, since it can be easily constructed from Theorem 7.2 and Algorithm I (see Section 5.1). Furthermore, The above theorem has an interesting corollary.

**Corollary 7.3.** Given  $n$ , there exist  $k$  disjoint  $d$ -sequences  $A_1, \dots, A_k$  such that for each  $i$ ,  $\nabla(A_i)$  consists only of  $m$  and 1 and the number of  $m$  is  $a$ .

Here  $k$  is sharp. Notice that  $[n]$  contains  $k$  disjoint sets of size  $d$ . In the remainder of this section, we are going to prove Theorem 7.2, and hence Theorem 7.1. Before that, we need some preparations. Firstly, we fix a configuration  $\Delta$  in  $\text{CONF}(a, b)$  with its characteristic sequence  $\{x_0, \dots, x_{b-1}\}$ .

**Definition 7.2.** Given a subset  $B \subseteq V(C_n)$ , its *differential set*  $D(B)$  is defined to be  $\{b_i - b_j \pmod n \mid \forall b_i, b_j \in B\}$ .

**Proposition 7.4.**  $\underline{i} \cap \underline{j} \neq \emptyset$  if and only if  $j - i \in D(V_\Delta)$ . Here the subtracting is calculated with modulo  $n$ .

*Proof.* Recall that  $V_\Delta = \{0, k_0, k_1, \dots, k_{d-2}\}$  is the vertex set of  $C_{0,\Delta}$ . Then  $\underline{i} \cap \underline{j} \neq \emptyset$  if and only if there exists a pair of indices  $p, q$  such that  $i + k_p = j + k_q$ , which is equivalent to  $j - i \in D(V_\Delta)$ .  $\square$

**Corollary 7.5.**  $\underline{i} \cap \underline{j} \neq \emptyset$  if and only if  $\underline{i+1} \cap \underline{j+1} \neq \emptyset$ .

**Proposition 7.6.**  $D(V_\Delta) = \{l_i + l_{i+1} + \dots + l_{i+s} \mid 1 \leq i \leq d, 0 \leq s < d\} \cup 0$ .

*Proof.*  $\forall x, y \in V_\Delta$ , if  $x = y$ , then  $x - y = 0$ ; otherwise we have:  $x = l_0 + l_1 + \dots + l_p$  and  $y = l_0 + l_1 + \dots + l_q$  for some  $0 < p, q < d$  where  $p \neq q$ . If  $p > q$ , the  $x - y = l_{p+1} + \dots + l_q$ . Otherwise from  $(l_0 + l_1 + \dots + l_{d-1}) = n$  we have

$$\begin{aligned} x - y &\equiv x + n - y \\ &= (l_0 + l_1 + \dots + l_p) + (l_0 + l_1 + \dots + l_{d-1}) - (l_0 + l_1 + \dots + l_q) \\ &= (l_0 + l_1 + \dots + l_p) + (l_{q+1} + \dots + l_{d-1}) \\ &= l_{q+1} + \dots + l_{d-1} + l_0 + l_1 + \dots + l_p. \end{aligned}$$

$\square$

For the configuration  $\Delta$ , recall that  $\mu_j$  is defined in Section 3 as

$$\mu_j = \min_{0 \leq i \leq b-1} \{x_i + x_{i+1} + \dots + x_{i+j-1}\},$$

for  $1 \leq j \leq b$  and  $\mu_{-1} = \mu_0 = 0$ . Similarly we define  $\xi_j$  as

$$\xi_j = \max_{0 \leq i \leq b-1} \{x_i + x_{i+1} + \dots + x_{i+j}\}$$

for  $0 \leq j \leq b-1$  and  $\nu_b = a-1$ . Here we let  $\nu_{b-1} = a$  and  $\nu_b = a-1$  to satisfy the boundary condition in the following proposition.

**Proposition 7.7.**  $D(V_\Delta) = \{p_j m + j \mid 0 \leq j \leq b, \mu_{j-1} \leq p_j \leq \xi_j\}$

*Proof.* The boundary cases can be verified directly and the other cases are following from proposition (7.6) by considering the number of 1s in the expressions of the elements in  $D(V_\Delta)$ .  $\square$

Now we are going to consider the case when  $\Delta$  is the regular configuration in  $\text{Shift}(n, m)$ . The following theorem is a main step in the proof of Theorem 7.1.

**Theorem 7.8.** *For the regular configuration  $\Delta$  in  $\text{CONF}(a, b)$ ,  $\underline{0} \cap \underline{q(m-1)} = \emptyset$  for  $1 \leq q \leq k-1$ .*

*Proof.* We will prove the theorem by contradiction. By assumption,  $\underline{i}$  is the cycle  $C_{i,\Delta}$  for the regular configuration  $\Delta$  in  $\text{CONF}(n, m)$ . If the theorem fails, then there exists one integer  $q \in [1, k-1]$  such that  $\underline{0} \cap \underline{q(m-1)} \neq \emptyset$ .

From Proposition 7.4,  $q(m-1) \in D(V_\Delta)$ . Together with Proposition 7.7, this implies the following equation has a solution for the variables  $j$  and  $q$  such that  $0 \leq j \leq b$  and  $1 \leq q \leq k-1$ .

$$q(m-1) \equiv p_j m + j \pmod{n}. \quad (18)$$

Let

$$r = \lfloor \frac{q(m-1)}{am+b} \rfloor.$$

Since  $p_j m + j < n$  from the definition, Equation (18) can be simplified as:

$$q(m-1) = p_j m + j + r(am+b). \quad (19)$$

The above equation can be further simplified as:

$$m = \frac{q + rb + j}{q - p_j - ra} \quad (20)$$

Now we are going to deduce a contradiction from the assumption that the above equation has an integer solution for the variables  $j$  and  $q$ . To this end, we use the following claim, which will be proven later as Proposition 7.9.

**Claim:**  $q + rb + j < 2m$  for  $0 \leq j \leq b$ ,  $1 \leq q \leq k-1$ .

By this claim, Equation (20) has integer solutions if and only if the following two equations have integer solutions:

$$q + rb + j = m \quad (21)$$

$$q - p_j - ra = 1 \quad (22)$$

By eliminating  $q$  from the above equations we have:

$$m = 1 + rb + j + p_j + ra = 1 + r(a + b) + j + p_j. \quad (23)$$

On the other hand, we can solve  $q$  from Equation (21) to get:  $q = m - rb - j$ . Together with  $q \leq k - 1$ , we have:

$$k \geq 1 + m - rb - j.$$

Since

$$k = \lfloor \frac{am + b}{a + b} \rfloor,$$

we know,

$$\begin{aligned} k(a + b) &\leq am + b \\ \Rightarrow (1 + m - rb - j)(a + b) &\leq am + b \\ \Rightarrow a + bm &\leq rb(a + b) + j(a + b) \\ \Rightarrow m &\leq r(a + b) + (1 + \frac{a}{b})j - \frac{a}{b}. \end{aligned}$$

Together with Equation(23):

$$\begin{aligned} 1 + r(a + b) + j + p_j &\leq r(a + b) + (1 + \frac{a}{b})j - \frac{a}{b} \\ \Rightarrow 1 + p_j &\leq \frac{a}{b}(j - 1) \\ \Rightarrow 1 + \mu_{j-1} &\leq \frac{a}{b}(j - 1) \end{aligned}$$

where the last step comes from the fact that  $\mu_{j-1} \leq p_j$ . Therefore if the theorem fails, then there must exist an  $j \in [0, b]$  such that  $1 + \mu_{j-1} \leq \frac{a}{b}(j - 1)$ . But from Lemma 3.1,  $1 + \mu_{j-1} > \frac{a}{b}(j - 1)$  for all  $0 \leq j \leq b$  since  $\Delta$  is a regular configuration. Thus we get a contradiction, which completes the proof of this theorem under the assumption of the claim.  $\square$

Now we are going to prove the claim to complete the proof of Theorem 7.8.

**Proposition 7.9.**  $q + rb + j < 2m$  for  $0 \leq j \leq b$ ,  $1 \leq q \leq k - 1$ .

*Proof.* We will prove it by contradiction. If not, we have  $q + rb + j \geq 2m$ . Together with the assumption  $j \leq b$ , it implies  $q + (r + 1)b \geq 2m$ . Since  $q \leq k - 1$  and

$$r = \lfloor \frac{q(m - 1)}{am + b} \rfloor,$$

we have:

$$(k-1) + (1 + \frac{(k-1)(m-1)}{am+b}) \geq 2m \quad (24)$$

By using the fact that

$$k = \lfloor \frac{am+b}{a+b} \rfloor,$$

we get:

$$(\frac{am+b}{a+b} - 1) + (1 + \frac{(\frac{am+b}{a+b} - 1)(m-1)}{am+b})b \geq 2m. \quad (25)$$

It can be further simplified as:

$$a(a+b)bm + (ab+b^2)b \geq a(a+b)m^2 + (a+b)(a+2b)m. \quad (26)$$

By dividing  $a+b$  from both sides, we obtain:

$$\begin{aligned} abm + b^2 &\geq am^2 + am + 2bm \\ \Rightarrow b^2 &\geq (am + a + 2b - ab)m \\ \Rightarrow b^2 &\geq (a + 2b)m. \end{aligned}$$

This is a contradiction since  $a \geq 0$ ,  $0 \leq b < m$  and  $a+b \neq 0$ .  $\square$

The last step in this section is to prove Theorem 7.2 with Theorem 7.8, which is relatively easy.

*The proof of Theorem 7.2* If the theorem fails, then two cycles in  $\mathcal{C}$  are not disjoint, which means  $\underline{0} \cap \underline{q(m-1)} \neq \emptyset$  for some  $1 \leq q \leq k-1$ , a contradiction to Theorem 7.8 since  $\Delta$  is a regular configuration.

## 8 Conclusions

In this paper, we give a relatively new definition of regular configurations, which unifies the “regularity” defined in many models. Some properties of regular configurations are discussed, including their constructions and the symmetry groups.

Regular configurations, or balanced words as it called in symbolic dynamics (see Section 6), are showed to optimize a number of quantities in words and ergodic theory [12]. In this paper we extend this to graph theory. They are used to obtain a bound of the cycles packing number for shift graphs. In a forthcoming paper [20], a polynomial algorithm is proposed to calculate  $\nu_0(\text{Shift}(n, m))$  while to calculate  $\nu_0(G)$  is **NP**-hard for general graph  $G$ .

A model not covered in this paper is the Kawasaki-Ising model, which is studied in [6]. There regular configurations are characterized as the ground states in this model.

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